2. FUNCTIONS

FUNCTION *Let A and B be two nonempty sets. Then, a rule f which associates to each element* $x \in A$*, a unique element, denoted by* $f(x)$ *of B, is called a function from A to B and we write,*

$$f : A \in B.$$

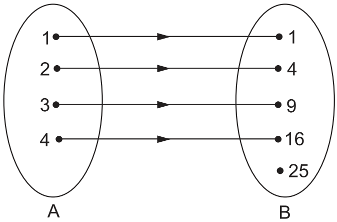
$f(x)$ *is called the* ***image*** *of x, while x is called the* ***pre-image*** *of* $f(x)$*.*

Domain, Codomain and Range of a Function

Let $f : A \to B$. Then, A is called the *domain* of f and B is called the *codomain* of f.

And, $f(A) = \{f(x) : x \in A\}$ is called the *range* of f.

Example 1 Let $A = \{1, 2, 3, 4\}$ and $B = \{1, 4, 9, 16, 25\}$.



Consider the rule $f : A \to B : f(x) = x^2 \;\forall\; x \in A$.

Then, each element in A has its unique image in B.

So, f is a function from A to B.

$$f(1) = 1^2 = 1, \;\; f(2) = 2^2 = 4, \;\; f(3) = 3^2 = 9, \;\; f(4) = 4^2 = 16.$$

Dom $(f) = \{1, 2, 3, 4\} = A$, codomain $(f) = \{1, 4, 9, 16, 25\} = B$ and range $(f) = \{1, 4, 9, 16\}$.

Clearly, $25 \in B$ does not have its pre-image in A.

Example 2 Let N be the set of all natural numbers.

Let $f : N \to N : f(x) = 2x \;\forall\; x \in N$.

Then, every element in N has its unique image in N.

So, f is a function from N to N.

Clearly, $f(1) = 2, \;\; f(2) = 4, \;\; f(3) = 6$ ..., and so on.

Dom $(f) = N$, codomain $(f) = N$,

range $(f) = \{2, 4, 6, 8, 10, ...\}$.

Various Types of Functions

MANY-ONE FUNCTION *A function $f : A \to B$ is said to be many-one if two or more than two elements in A have the same image in B.*

Example Let $A = \{-1, 1, 2, 3\}$ and $B = \{1, 4, 9\}$.

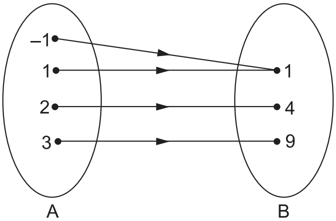
Let $f : A \to B : f(x) = x^2 \; \forall \; x \in A$.

Then, each element in A has a unique image under f in B.

$\therefore$ f is a function from A to B such that

$f(-1) = (-1)^2 = 1$; $f(1) = 1^2 = 1$;

$f(2) = 2^2 = 4$ and $f(3) = 3^2 = 9$.



Clearly, two different elements, namely –1 and 1, have the same image $1 \in B$.

Hence, f is many-one.

One-One or Injective Function

A function $f : A \to B$ is said to be one-one if distinct elements in A have distinct images in B.

f is one-one when $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$.

Example Let N be the set of all natural numbers.

Let $f : N \to N : f(x) = 2x \; \forall \; x \in N$.

Then, $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2$

$\Rightarrow x_1 = x_2$.

$\therefore$ f is one-one.

Onto or Surjective Function

A function $f : A \to B$ is said to be onto if every element in B has at least one pre-image in A.

Thus, if f is onto then for each $y \in B$ $\exists$ at least one element $x \in A$ such that $y = f(x)$.

Also, f is onto $\Leftrightarrow$ range $(f) = B$.

Example Let N be the set of all natural numbers and let E be the set of all even natural numbers.

Let $f : N \to E : f(x) = 2x \; \forall \; x \in N$.

Then, $y = 2x \Rightarrow x = \frac{1}{2}y$.

Thus, for each $y \in E$ there exists $\frac{1}{2}y \in N$ such that

$$f\left(\frac{1}{2}y\right) = \left(2 \times \frac{1}{2}y\right) = y.$$

$\therefore$ f is onto.

INTO FUNCTION *A function $f : A \to B$ is said to be into if there exists even a single element in B having no pre-image in A.*

Clearly, f is into $\Leftrightarrow$ range $(f) \subset B$.

Example Let $A = \{2, 3, 5, 7\}$ and $B = \{0, 1, 3, 5, 7\}$.

Let $f : A \to B : f(x) = (x - 2)$. Then,

$f(2) = (2 - 2) = 0$, $f(3) = (3 - 2) = 1$, $f(5) = (5 - 2) = 3$ and $f(7) = (7 - 2) = 5$.

Thus, every element in A has a unique image in B.

Now, $\exists\ 7 \in B$ having no pre-image in A.

$\therefore$ f is into.

Note that range $(f) = \{0, 1, 3, 5\} \subset B$.

Bijective Function

A one-one onto function is said to be bijective or a one-to-one correspondence.

CONSTANT FUNCTION *A function $f : A \to B$ is called a constant function if every element of A has the same image in B.*

Example Let $A = \{1, 2, 3\}$ and $B = \{5, 7, 9\}$.

Let $f : A \to B : f(x) = 5$ for all $x \in A$.

Clearly, every element in A has the same image.

So, f is a constant function.

REMARK The range of a constant function is a singleton set.

IDENTITY FUNCTION *The function $I_A : A \to A : I_A(x) = x\ \forall\ x \in A$ is called an identity function on A.*

Domain $(I_A) = A$ and range $(I_A) = A$.

EQUAL FUNCTIONS *Two functions f and g having the same domain D are said to be equal if $f(x) = g(x)\ \forall\ x \in D$.*

SOLVED EXAMPLES

EXAMPLE 1 *Let $f : N \to N : f(x) = 2x$ for all $x \in N$.*
Show that f is one-one and into.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Let $y = 2x$. Then, $x = \frac{y}{2}$.

If we put $y = 3$ then $x = \frac{3}{2} \notin N$.

Thus, $3 \in N$ has no pre-image in N.

$\therefore$ f is into.

Hence, f is one-one and into.

EXAMPLE 2 *Show that the function $f : R \to R : f(x) = x^2$ is neither one-one nor onto.*

SOLUTION We have $f(-1) = (-1)^2 = 1$ and $f(1) = 1^2 = 1$.

Thus, two different elements in R have the same image.

$\therefore$ f is not one-one.

If we consider –1 in the codomain R, then it is clear that there is no element in R whose image is –1.

$\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 3 *Show that the function $f : R \to R : f(x) = |x|$ is neither one-one nor onto.*

SOLUTION We have $f(-1) = |-1| = 1$ and $f(1) = |1| = 1$.

Thus, two different elements in R have the same image.

$\therefore$ f is not one-one.

If we consider –1 in the codomain R, then it is clear that there is no real number x whose modulus is –1.

Thus, $-1 \in R$ has no pre-image in R.

$\therefore$ f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 4 *For any real number x, we define*

$[x]$ = *greatest integer less than or equal to x.*

Prove that the greatest integer function $f : R \to R : f(x) = [x]$ is neither one-one nor onto.

SOLUTION Clearly, $[1.2] = 1$ and $[1.3] = 1$.

$\therefore$ $f(1.2) = 1$ and $f(1.3) = 1$.

Thus, two different real numbers have the same image.

$\therefore$ f is not one-one.

Clearly, there is no real number x such that

$f(x) = [x] = 1.1.$

So, f is not onto.

Hence, f is neither one-one nor onto.

EXAMPLE 5 *Let R_0 be the set of all nonzero real numbers.*
Show that $f : R_0 \to R_0 : f(x) = \frac{1}{x}$ is a one-one onto function.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Again, $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$.

Now, if y is a nonzero real number, then $x = \left(\frac{1}{y}\right)$ is a nonzero real number such that $f\left(\frac{1}{y}\right) = y$.

Thus, each y in R_0 has its pre-image in R_0.

So, f is onto.

Hence, f is one-one onto.

EXAMPLE 6 *Show that the function $f : R \to R : f(x) = x^3$ is one-one and onto.*

SOLUTION We have

$$\begin{aligned}
f(x_1) = f(x_2) &\Rightarrow x_1^3 = x_2^3 \\
&\Rightarrow (x_1^3 - x_2^3) = 0 \\
&\Rightarrow (x_1 - x_2)(x_1^2 + x_1x_2 + x_2^2) = 0 \\
&\Rightarrow (x_1 - x_2)\left[\left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2\right] = 0 \\
&\Rightarrow (x_1 - x_2) = 0 \quad \left[\because \left(x_1 + \frac{x_2}{2}\right)^2 + \frac{3}{4}x_2^2 \neq 0\right] \\
&\Rightarrow x_1 = x_2.
\end{aligned}$$

$\therefore$ f is one-one.

Let $y \in R$ and let $y = x^3$. Then, $x = y^{1/3} \in R$.

Thus, for each y in the codomain R there exists $y^{1/3}$ in R such that $f(y^{1/3}) = (y^{1/3})^3 = y$.

$\therefore$ f is onto.

Hence, f is one-one onto.

EXAMPLE 7 *Show that the function $f : R \to R : f(x) = 3 - 4x$ is one-one onto and hence bijective.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow 3 - 4x_1 = 3 - 4x_2$$

$\Rightarrow \quad -4x_1 = -4x_2 \Rightarrow x_1 = x_2.$

$\therefore \quad f$ is one-one.

Now, let $y = 3 - 4x$. Then, $x = \dfrac{(3-y)}{4}$.

Thus, for each $y \in R$ (codomain of f), there exists $x = \dfrac{(3-y)}{4} \in R$ such that $f(x) = f\left(\dfrac{3-y}{4}\right) = \left\{3 - 4 \cdot \dfrac{(3-y)}{4}\right\} = 3 - (3-y) = y.$

This shows that every element in codomain of f has its pre-image in dom (f).

$\therefore \quad f$ is onto.

Hence, the given function is bijective.

EXAMPLE 8 *Show that the function $f: N \rightarrow N$, defined by*

$$f(x) = \begin{cases} x+1, & \text{if } x \text{ is odd} \\ x-1, & \text{if } x \text{ is even} \end{cases}$$

is one-one and onto. **[CBSE 2012]**

SOLUTION Suppose $f(x_1) = f(x_2)$.

Case 1 *When x_1 is odd and x_2 is even*

In this case, $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 - 1$

$\Rightarrow x_2 - x_1 = 2.$

This is a contradiction, since the difference between an odd integer and an even integer can never be 2.

Thus, in this case, $f(x_1) \neq f(x_2)$.

Similarly, when x_1 is even and x_2 is odd, then $f(x_1) \neq f(x_2)$.

Case 2 *When x_1 and x_2 are both odd*

In this case, $f(x_1) = f(x_2) \Rightarrow x_1 + 1 = x_2 + 1$

$\Rightarrow x_1 = x_2.$

$\therefore f$ is one-one.

Case 3 *When x_1 and x_2 are both even*

In this case, $f(x_1) = f(x_2) \Rightarrow x_1 - 1 = x_2 - 1$

$\Rightarrow x_1 = x_2.$

$\therefore \quad f$ is one-one.

In order to show that f is onto, let $y \in N$ (the codomain).

Case 1 *When y is odd*

In this case, $(y+1)$ is even.

$\therefore \quad f(y+1) = (y+1) - 1 = y.$

Case 2 *When y is even*

In this case, $(y-1)$ is odd.

$\therefore \quad f(y-1) = y - 1 + 1 = y.$

Thus, each $y \in N$ (codomain of f) has its pre-image in dom (f).

$\therefore$ f is onto.

Hence, f is one-one onto.

EXAMPLE 9 *Show that $f : N \to N$, defined by*

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{if } n \text{ is odd} \\ \frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

is a many-one onto function. **[CBSE 2012C]**

SOLUTION We have

$$f(1) = \frac{(1+1)}{2} = \frac{2}{2} = 1 \quad \text{and} \quad f(2) = \frac{2}{2} = 1.$$

Thus, $f(1) = f(2)$ while $1 \neq 2$.

$\therefore$ f is many-one.

In order to show that f is onto, consider an arbitrary element $n \in N$.

If n is odd then $(2n - 1)$ is odd, and

$$f(2n-1) = \frac{(2n-1+1)}{2} = \frac{2n}{2} = n.$$

If n is even then $2n$ is even and

$$f(2n) = \frac{2n}{2} = n.$$

Thus, for each $n \in N$ (whether even or odd) there exists its pre-image in N.

$\therefore$ f is onto.

Hence, f is many-one onto.

EXAMPLE 10 *Show that the signum function $f : R \to R$, defined by*

$$f(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}$$

is neither one-one nor onto.

SOLUTION Clearly, $f(2) = 1$ and $f(3) = 1$.

Thus, $f(2) = f(3)$ while $2 \neq 3$.

$\therefore$ f is not one-one.

Range (f) $= \{1, \ 0, \ -1\} \subset R$.

So, f is into.

Hence, f is neither one-one nor onto.

EXAMPLE 11 *Let* $A = R - \{3\}$ *and* $B = R - \{1\}$.
Let $f : A \to B : f(x) = \dfrac{x-2}{x-3}$ *for all values of* $x \in A$.
Show that f is one-one and onto. **[CBSE 2012]**

SOLUTION f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1 - 2}{x_1 - 3} = \frac{x_2 - 2}{x_2 - 3}$$
$$\Rightarrow (x_1 - 2)(x_2 - 3) = (x_1 - 3)(x_2 - 2)$$
$$\Rightarrow x_1x_2 - 3x_1 - 2x_2 + 6 = x_1x_2 - 2x_1 - 3x_2 + 6$$
$$\Rightarrow x_1 = x_2.$$

Let $y \in B$ such that $y = \dfrac{x-2}{x-3}$.

Then, $(x-3)\, y = (x-2) \Rightarrow x = \dfrac{(3y-2)}{(y-1)}$.

Clearly, x is defined when $y \neq 1$.

Also, $x = 3$ will give us $1 = 0$, which is false.

$\therefore \quad x \neq 3$.

$$\text{And, } f(x) = \frac{\left(\dfrac{3y-2}{y-1} - 2\right)}{\left(\dfrac{3y-2}{y-1} - 3\right)} = y.$$

Thus, for each $y \in B$, there exists $x \in A$ such that $f(x) = y$.

$\therefore \quad f$ is onto.

Hence, f is one-one onto.

EXAMPLE 12 *Let A and B be two nonempty sets. Show that the function* $f : (A \times B) \to (B \times A) : f(a, b) = (b, a)$ *is a bijective function.*

SOLUTION f is one-one, since

$$f(a_1, b_1) = f(a_2, b_2) \Rightarrow (b_1, a_1) = (b_2, a_2)$$
$$\Rightarrow a_1 = a_2 \text{ and } b_1 = b_2$$
$$\Rightarrow (a_1, b_1) = (a_2, b_2).$$

In order to show that f is onto, let (b, a) be an arbitrary element of $(B \times A)$.

Then, $(b, a) \in (B \times A) \Rightarrow b \in B$ and $a \in A$
$\Rightarrow (a, b) \in (A \times B)$.

Thus, for each $(b, a) \in (B \times A)$, there exists $(a, b) \in A \times B$ such that $f(a, b) = (b, a)$.

$\therefore \quad f$ is onto.

Thus, f is one-one onto and hence bijective.

EXAMPLE 13 *Find the domain and range of the real function*

$$f(x) = \sqrt{9 - x^2}.$$

SOLUTION It is clear that $f(x) = \sqrt{9 - x^2}$ is not defined when $(9 - x^2) < 0$, i.e., when $x^2 > 9$, i.e., when $x > 3$ or $x < -3$.

$\therefore$ dom $(f) = \{x \in R : -3 \le x \le 3\}$.

Also, $y = \sqrt{9 - x^2} \Rightarrow y^2 = (9 - x^2)$

$\Rightarrow x = \sqrt{9 - y^2}$.

Clearly, x is not defined when $(9 - y^2) < 0$.

But, $(9 - y^2) < 0 \Rightarrow y^2 > 9$

$\Rightarrow y > 3$ or $y < -3$

$\therefore$ range $(f) = \{y \in R : -3 \le y \le 3\}$.

EXAMPLE 14 *Find the domain and range of the real function, defined by*

$$f(x) = \frac{1}{(1 - x^2)}.$$

SOLUTION Clearly, $\frac{1}{(1 - x^2)}$ is not defined when $1 - x^2 = 0$,

i.e., when $x = \pm 1$.

$\therefore$ dom $(f) = R - \{-1, 1\}$.

Also, $y = \frac{1}{(1 - x^2)} \Rightarrow (1 - x^2) = \frac{1}{y} \Rightarrow x = \sqrt{1 - \frac{1}{y}}$.

Clearly, x is not defined when $\left(1 - \frac{1}{y}\right) < 0$ or $1 < \frac{1}{y}$ or $y < 1$.

$\therefore$ range $(f) = R - \{y \in R : y < 1\} = \{y \in R : y \ge 1\}$.

EXAMPLE 15 *Consider a function* $f : X \to Y$ *and define a relation* R *in* X *by* $R = \{(a, b) : f(a) = f(b)\}$. *Show that* R *is an equivalence relation.*

SOLUTION Here, R satisfies the following properties:

(i) *Reflexivity*
Let $a \in X$. Then,

$f(a) = f(a) \Rightarrow (a, a) \in R$.

$\therefore$ R is reflexive.

(ii) *Symmetry*
Let $(a, b) \in R$. Then,

$(a, b) \in R \Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a) \Rightarrow (b, a) \in R$.

$\therefore$ R is symmetric.

(iii) *Transitivity*

Let $(a, b) \in R$ and $(b, c) \in R$. Then,

$$(a, b) \in R, (b, c) \in R$$

$$\Rightarrow f(a) = f(b) \text{ and } f(b) = f(c)$$

$$\Rightarrow f(a) = f(c)$$

$$\Rightarrow (a, c) \in R.$$

$\therefore$ R is transitive.

Hence, R is an equivalence relation.

EXERCISE 2A

1. Define a function. What do you mean by the domain and range of a function? Give examples.
2. Define each of the following:
   (i) injective function (ii) surjective function
   (iii) bijective function (iv) many-one function
   (v) into function
   Give an example of each type of functions.
3. Give an example of a function which is
   (i) one-one but not onto (ii) one-one and onto
   (iii) neither one-one nor onto (iv) onto but not one-one.
4. Let $f : R \to R$ be defined by
   $$f(x) = \begin{cases} 2x + 3, & \text{when } x < -2 \\ x^2 - 2, & \text{when } -2 \le x \le 3 \\ 3x - 1, & \text{when } x > 3. \end{cases}$$
   Find (i) $f(2)$ (ii) $f(4)$ (iii) $f(-1)$ (iv) $f(-3)$.
5. Show that the function $f : R \to R : f(x) = 1 + x^2$ is many-one into.
6. Show that the function $f : R \to R : f(x) = x^4$ is many-one and into.
7. Show that the function $f : R \to R : f(x) = x^5$ is one-one and onto.
8. Let $f : \left[0, \frac{\pi}{2}\right] \to R : f(x) = \sin x$ and $g : \left[0, \frac{\pi}{2}\right] \to R : g(x) = \cos x$. Show that each one of f and g is one-one but $(f + g)$ is not one-one.
9. Show that the function
   (i) $f : N \to N : f(x) = x^2$ is one-one into.
   (ii) $f : Z \to Z : f(x) = x^2$ is many-one into.

10. Show that the function
   (i) $f: N \to N: f(x) = x^3$ is one-one into
   (ii) $f: Z \to Z: f(x) = x^3$ is one-one into
11. Show that the function $f: R \to R: f(x) = \sin x$ is neither one-one nor onto.
12. Prove that the function $f: N \to N: f(n) = (n^2 + n + 1)$ is one-one but not onto.
13. Show that the function $f: N \to Z$, defined by

$$f(n) = \begin{cases} \frac{1}{2}(n-1), & \text{when } n \text{ is odd} \\ -\frac{1}{2}n, & \text{when } n \text{ is even} \end{cases}$$

   is both one-one and onto.
14. Find the domain and range of the function
   $f: R \to R: f(x) = x^2 + 1.$
15. Which of the following relations are functions? Give reasons. In case of a function, find its domain and range.
   (i) $f = \{(-1, 2), (1, 8), (2, 11), (3, 14)\}$
   (ii) $g = \{(1, 1), (1, -1), (4, 2), (9, 3), (16, 4)\}$
   (iii) $h = \{(a, b), (b, c), (c, b), (d, c)\}$
16. Find the domain and range of the real function, defined by $f(x) = \dfrac{x^2}{(1 + x^2)}$.
   Show that f is many-one.
17. Show that the function

$$f: R \to R: f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational} \end{cases}$$

   is many-one into.
   Find (i) $f\left(\frac{1}{2}\right)$ (ii) $f(\sqrt{2})$ (iii) $f(\pi)$ (iv) $f(2 + \sqrt{3})$.

ANSWERS (EXERCISE 2A)

4. (i) 2 (ii) 11 (iii) –1 (iv) –3
14. dom $(f) = R$ and range $(f) = \{y \in R : y \geq 1\}$
15. (i) f is a function, dom $(f) = \{-1, 1, 2, 3\}$ and range $(f) = \{2, 8, 11, 14\}$
(ii) g is not a function
(iii) h is a function, dom $(h) = \{a, b, c, d\}$ and range $(h) = \{b, c\}$
17. (i) 1 (ii) –1 (iii) –1 (iv) –1

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 2A)

2. (i) $f: N \to N: f(x) = 2x$ is an injective function, as
$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.

(ii) Let $A = \{1, -1, 2, 3\}$ and $B = \{1, 4, 9\}$.
Then, $f: A \to B: f(x) = x^2$ is surjective, since each element of B has at least one pre-image in A.

(iii) Let E be the set of all even natural numbers.
Then, $f: N \to E: f(x) = 2x$ is one-one and onto.
f is one-one, since
$$f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2.$$
f is onto, since for each $y \in E$, there exists $\frac{1}{2}y \in N$ such that $f\left(\frac{1}{2}y\right) = y$.

(iv) Example given in (ii) is many-one.

(v) Let $A = \{1, 2, 3\}$ and $B = \{1, 4, 9, 16\}$.
Then, $f: A \to B: f(x) = x^2$ is an into function, since range $(f) = \{1, 4, 9\} \subset B$.

5. $f(-1) = 2 = f(1)$. So, f is many-one.
$-1 \in R$ has no pre-image in R. So, f is into.

6. $f(-1) = f(1) = 1$. So, f is many-one.
$-1 \in R$ has no pre-image in R. So, f is into.

7. $f(x_1) = f(x_2) \Rightarrow x_1^5 = x_2^5 \Rightarrow x_1 = x_2$. So, f is one-one
for each $y \in R \; \exists \; y^{1/5} \in R$ s.t. $f(y^{1/5}) = y$.

8. If $x_1 \neq x_2$ and $x_1, x_2 \in \left[0, \frac{\pi}{2}\right]$ then $\sin x_1 \neq \sin x_2$ and $\cos x_1 \neq \cos x_2$.
$\therefore$ f is one-one and g is one-one.
But $(f + g)(x) = f(x) + g(x) = \sin x + \cos x$.
$\therefore$ $(f + g)(0) + (\sin 0 + \cos 0) = 1$ and $(f + g)\left(\frac{\pi}{2}\right) = \left(\sin \frac{\pi}{2} + \cos \frac{\pi}{2}\right) = 1$.

9. (i) $f(a) = f(b) \Rightarrow a^2 = b^2 \Rightarrow a = b \quad [\because a, b \in N]$
$\therefore$ f is one-one.
Clearly, $2 \in N$ [codomain (f)] has no pre-image in N.
$\therefore$ f is into.

(ii) $f(-1) = f(1) = 1$. So, f is many-one.
$2 \in N$ [codomain (f)] has no pre-image in N.
$\therefore$ f is into.

10. (i) $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2 \quad [\because x_1, x_2 \in N]$.
$\therefore$ f is one-one.
$2 \in N$. But, $2^{1/3} \notin N$. So, f is into.

(ii) $f(x_1) = f(x_2) \Rightarrow x_1^3 = x_2^3 \Rightarrow x_1 = x_2 \quad [\because x_1, x_2 \in Z]$
$2 \in Z$. But, $2^{1/3} \notin Z$. So, f is into.

11. We know that $\sin(0) = 0$ and $\sin(\pi) = 0$.

Thus, 0 and π have the same image.

So, f is many-one.

Range $(f) = [-1, 1] \subset R$. Hence, f is into.

So, f is neither one-one nor onto.

12. $f(n_1) = f(n_2) \Rightarrow n_1^2 + n_1 + 1 = n_2^2 + n_2 + 1$

$$\Rightarrow (n_1^2 - n_2^2) + (n_1 - n_2) = 0$$
$$\Rightarrow (n_1 - n_2)(n_1 + n_2 + 1) = 0$$
$$\Rightarrow n_1 - n_2 = 0 \Rightarrow n_1 = n_2.$$

$\therefore$ f is one-one.

But, $f(n) = 1 \Rightarrow n^2 + n + 1 = 1 \Rightarrow n^2 + n = 0$

$$\Rightarrow n(n+1) = 0 \Rightarrow n = 0 \text{ or } n = -1.$$

And, none of 0 and –1 is a natural number.

Thus, $1 \in N$ has no pre-image in N.

$\therefore$ f is into.

14. Dom $(f) = R$. Also, $y = x^2 + 1 \Rightarrow x = \sqrt{y-1}$.

x is defined when $y - 1 \geq 0$, i.e., $y \geq 1$.

$\therefore$ range $(f) = \{y \in R : y \geq 1\}$.

15. g is not a function, since 1 has two images under g.

16. When x is real, $1 + x^2 \neq 0$. So, dom $(f) = R$.

$$y = \frac{x^2}{(1+x^2)} \Rightarrow x^2(1-y) = y \Rightarrow x = \sqrt{\frac{y}{1-y}}.$$

For x to be real, $\dfrac{y}{(1-y)} \geq 0$ and $(1-y) \neq 0$.

$\therefore$ range $(f) = \{y \in R : 0 \leq y < 1\}$.

Also, 1 and –1 have the same image $\left(\frac{1}{2}\right)$.

Composition of Functions

Let $f : A \to B$ and $g : B \to C$ be two given functions. Then, the composition of f and g, denoted by $g \circ f$ is the function, defined by

$$(g \circ f) : A \to C : (g \circ f)(x) = g\{f(x)\} \ \forall \ x \in A.$$

Clearly, dom $(g \circ f)$ = dom (f).

Also, $g \circ f$ is defined only when range $(f) \subseteq$ dom (g).

REMARK $(f \circ g)$ is defined only when range $(g) \subseteq$ dom (f).
And, dom $(f \circ g)$ = dom (g).

SOLVED EXAMPLES

EXAMPLE 1 *Let* $f : \{1, 3, 4\} \to \{1, 2, 5\}$ *and* $g : \{1, 2, 5\} \to \{1, 3\}$ *be defined as* $f = \{(1, 2), (3, 5), (4, 1)\}$ *and* $g = \{(1, 3), (2, 3), (5, 1)\}$. *Find* $(g \circ f)$ *and* $(f \circ g)$.

SOLUTION Here range $(f) = \{1, 2, 5\}$ and dom $(g) = \{1, 2, 5\}$.

Clearly, range $(f) \subseteq$ dom (g).

$\therefore$ $(g \circ f)$ is defined and dom $(g \circ f) =$ dom $(f) = \{1, 3, 4\}$.

Now, $(g \circ f)(1) = g\{f(1)\} = g(2) = 3$;

$(g \circ f)(3) = g\{f(3)\} = g(5) = 1$;

$(g \circ f)(4) = g\{f(4)\} = g(1) = 3$.

Hence, $(g \circ f) = \{(1, 3), (3, 1), (4, 3)\}$.

Again, range $(g) = \{1, 3\}$ and dom $(f) = \{1, 3, 4\}$.

Clearly, range $(g) \subseteq$ dom (f).

$\therefore$ $(f \circ g)$ is defined and dom $(f \circ g) =$ dom $(g) = \{1, 2, 5\}$.

Now, $(f \circ g)(1) = f\{g(1)\} = f(3) = 5$;

$(f \circ g)(2) = f\{g(2)\} = f(3) = 5$;

$(f \circ g)(5) = f\{g(5)\} = f(1) = 2$.

Hence, $(f \circ g) = \{(1, 5), (2, 5), (5, 2)\}$.

EXAMPLE 2 *Let R be the set of all real numbers. Let* $f : R \to R : f(x) = \cos x$ *and let* $g : R \to R : g(x) = 3x^2$. *Show that* $(g \circ f) \neq (f \circ g)$.

SOLUTION Let x be an arbitrary real number. Then,

$$(g \circ f)(x) = g\{f(x)\} = g(\cos x) = 3(\cos x)^2 = 3\cos^2 x.$$

$$(f \circ g)(x) = f\{g(x)\} = f(3x^2) = \cos(3x^2).$$

Taking $x = 0$, we have

$$(g \circ f)(0) = 3\cos^2 0 = (3 \times 1) = 3.$$

$$(f \circ g)(0) = \cos(3 \times 0) = \cos 0 = 1.$$

$\therefore$ $(g \circ f)(0) \neq (f \circ g)(0)$.

Hence, $g \circ f \neq f \circ g$.

EXAMPLE 3 *Let R be the set of all real numbers. Let* $f : R \to R : f(x) = \sin x$ *and* $g : R \to R : g(x) = x^2$. *Prove that* $g \circ f \neq f \circ g$.

SOLUTION Let x be an arbitrary real number. Then,

$$(g \circ f)(x) = g\{f(x)\} = g(\sin x) = (\sin x)^2.$$

$$(f \circ g)(x) = f\{g(x)\} = f(x^2) = \sin x^2.$$

Clearly, $(\sin x)^2 \neq \sin x^2$.

Hence, $g \circ f \neq f \circ g$.

EXAMPLE 4 *Let* $f : R \to R : f(x) = 8x^3$ *and* $g : R \to R : g(x) = x^{1/3}$. *Find* $(g \circ f)$ *and* $(f \circ g)$ *and show that* $g \circ f \neq f \circ g$.

SOLUTION Let $x \in R$. Then, we have

$$(g \circ f)(x) = g\{f(x)\} = g(8x^3) = (8x^3)^{1/3} = 2x.$$

$$(f \circ g)(x) = f\{g(x)\} = f(x^{1/3}) = 8(x^{1/3})^3 = 8x.$$

$\therefore \quad g \circ f \neq f \circ g.$

EXAMPLE 5 *Let* $f : R \to R : f(x) = (x^2 - 3x + 2)$, *find* $(f \circ f)(x)$.

SOLUTION We have

$$(f \circ f)(x) = f\{f(x)\} = f(x^2 - 3x + 2) = f(y), \text{ where } y = (x^2 - 3x + 2)$$
$$= (y^2 - 3y + 2)$$
$$= (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$
$$= (x^4 - 6x^3 + 10x^2 - 3x).$$

EXAMPLE 6 *If* $f : R \to R : f(x) = (3 - x^3)^{1/3}$, *show that* $(f \circ f)(x) = x$.

SOLUTION We have

$$(f \circ f)(x) = f\{f(x)\} = f(3 - x^3)^{1/3}$$
$$= f(y), \quad \text{where } y = (3 - x^3)^{1/3}$$
$$= (3 - y^3)^{1/3} = [3 - (3 - x^3)]^{1/3} \quad [\because \; y^3 = (3 - x^3)]$$
$$= (x^3)^{1/3} = x.$$

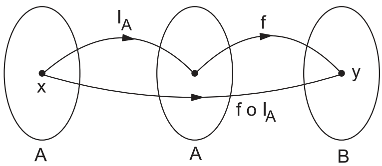
Hence, $(f \circ f)(x) = x$.

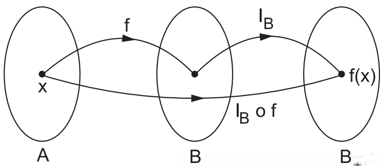
EXAMPLE 7 *Let* $f : A \to B$, *and let* I_A *and* I_B *be identity functions on* A *and* B *respectively. Prove that* $(f \circ I_A) = f$ *and* $(I_B \circ f) = f$.

SOLUTION Let $x \in A$ and let $f(x) = y$. Then,

$$(f \circ I_A)(x) = f\{I_A(x)\} = f(x) \qquad [\because \; I_A(x) = x].$$

$\therefore \quad (f \circ I_A) = f.$





And, $(I_B \circ f)(x) = I_B\{f(x)\}$
$= I_B(y)$ [$\because$ $f(x) = y$]
$= y$ [$\because$ $I_B(y) = y$]
$= f(x)$ [$\because$ $y = f(x)$].

$\therefore$ $(I_B \circ f) = f$.

Hence, $(f \circ I_A) = f$ and $(I_B \circ f) = f$.

EXAMPLE 8 (Associativity) *Let $f : A \to B$, $g : B \to C$ and $h : C \to D$. Then, prove that $(h \circ g) \circ f = h \circ (g \circ f)$.*

SOLUTION Let $x \in A$. Then,

$$\{(h \circ g) \circ f\}(x) = (h \circ g)\{f(x)\}$$
$$= h[g\{f(x)\}]$$
$$= h[(g \circ f)(x)]$$
$$= \{h \circ (g \circ f)\}(x).$$

$\therefore$ $(h \circ g) \circ f = h \circ (g \circ f)$.

EXAMPLE 9 *Let $f : Z \to Z : f(n) = 3n$ and let $g : Z \to Z$, defined by*

$$g(n) = \begin{cases} \frac{n}{3}, & \text{if } n \text{ is a multiple of } 3 \\ 0, & \text{if } n \text{ is not a multiple of } 3. \end{cases}$$

Show that $g \circ f = I_Z$ and $f \circ g \neq I_Z$.

SOLUTION Let n be an arbitrary element of Z. Then,

$$(g \circ f)(n) = g\{f(n)\}$$
$$= g(3n) = \frac{3n}{3} = n$$
$$= I_Z(n).$$

$\therefore$ $(g \circ f) = I_Z$.

Also, we have

$(f \circ g)(1) = f\{g(1)\}$
$= f(0)$ [$\because$ $g(1) = 0$]
$= (3 \times 0) = 0$ [$\because$ $f(n) = 3n$].

$I_Z(1) = 1$ [$\because$ $I_Z(x) = x \ \forall \ x \in Z$].

$\therefore$ $f \circ g \neq I_Z$.

EXAMPLE 10 *Let $A = R - \left\{\frac{3}{5}\right\}$ and $B = R - \left\{\frac{7}{5}\right\}$.*

Let $f : A \to B : f(x) = \frac{7x + 4}{5x - 3}$ and $g : B \to A : g(y) = \frac{3y + 4}{5y - 7}$.

Show that $(g \circ f) = I_A$ and $(f \circ g) = I_B$.

SOLUTION Let $x \in A$. Then,

$$(g \circ f)(x) = g[f(x)] = g\left(\frac{7x+4}{5x-3}\right)$$

$$= g(y), \text{ where } y = \frac{7x+4}{5x-3} \qquad \ldots \text{(i)}$$

$$= \frac{3y+4}{5y-7} = \frac{3\left(\frac{7x+4}{5x-3}\right)+4}{5\left(\frac{7x+4}{5x-3}\right)-7} \quad \text{[using (i)]}$$

$$= \frac{(21x+12+20x-12)}{(5x-3)} \times \frac{(5x-3)}{(35x+20-35x+21)}$$

$$= \frac{41x}{41} = x = I_A(x).$$

$\therefore \quad (g \circ f) = I_A$.

Again, let $y \in B$. Then,

$$(f \circ g)(y) = f[g(y)] = f\left(\frac{3y+4}{5y-7}\right)$$

$$= f(z), \text{ where } z = \frac{3y+4}{5y-7} \qquad \ldots \text{(ii)}$$

$$= \frac{7z+4}{5z-3} = \frac{7\left(\frac{3y+4}{5y-7}\right)+4}{5\left(\frac{3y+4}{5y-7}\right)-3}$$

$$= \frac{(21y+28+20y-28)}{(5y-7)} \times \frac{(5y-7)}{(15y+20-15y+21)}$$

$$= \frac{41y}{41} = y = I_B(y).$$

$\therefore \quad (f \circ g) = I_B$.

Hence, $(g \circ f) = I_A$ and $(f \circ g) = I_B$.

EXAMPLE 11 *Let $f: A \to B$ and $g: B \to A$ such that $(g \circ f) = I_A$. Show that f is one-one and g is onto.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow g\{f(x_1)\} = g\{f(x_2)\}$$

$$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$$

$$\Rightarrow I_A(x_1) = I_A(x_2)$$

$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

In order to show that g is onto, let $a \in A$ and let $f(a) = b \in B$.

Then, $g(b) = g[f(a)] = (g \circ f)(a)$

$= I_A(a)$ $[\because g \circ f = I_A]$.

Thus, for each $a \in A$, there exists $b \in B$ such that $g(b) = a$.

$\therefore$ g is onto.

EXERCISE 2B

1. Let $A = \{1, 2, 3, 4\}$. Let $f : A \to A$ and $g : A \to A$, defined by $f = \{(1, 4), (2, 1), (3, 3), (4, 2)\}$ and
$$g = \{(1, 3), (2, 1), (3, 2), (4, 4)\}.$$
Find (i) $g \circ f$ (ii) $f \circ g$ (iii) $f \circ f$.
2. Let $f : \{3, 9, 12\} \to \{1, 3, 4\}$ and $g : \{1, 3, 4, 5\} \to \{3, 9\}$ be defined as $f = \{(3, 1), (9, 3), (12, 4)\}$ and
$$g = \{(1, 3), (3, 3), (4, 9), (5, 9)\}.$$
Find (i) $(g \circ f)$ (ii) $(f \circ g)$.
3. Let $f : R \to R : f(x) = x^2$ and $g : R \to R : g(x) = (x + 1)$.
Show that $(g \circ f) \neq (f \circ g)$.
4. Let $f : R \to R : f(x) = (2x + 1)$ and $g : R \to R : g(x) = (x^2 - 2)$.
Write down the formulae for
(i) $(g \circ f)$ (ii) $(f \circ g)$ (iii) $(f \circ f)$ (iv) $(g \circ g)$.
5. Let $f : R \to R : f(x) = (x^2 + 3x + 1)$ and $g : R \to R : g(x) = (2x - 3)$. Write down the formulae for
(i) $g \circ f$ (ii) $f \circ g$ (iii) $g \circ g$.
6. Let $f : R \to R : f(x) = |x|$, prove that $f \circ f = f$.
7. Let $f : R \to R : f(x) = x^2$, $g : R \to R : g(x) = \tan x$ and $h : R \to R : h(x) = \log x$.
Find a formula for $h \circ (g \circ f)$.
Show that $[h \circ (g \circ f)] \sqrt{\frac{\pi}{4}} = 0$.
8. Let $f : R \to R : f(x) = (2x - 3)$ and $g : R \to R : g(x) = \frac{1}{2}(x + 3)$.
Show that $(f \circ g) = I_R = (g \circ f)$.
9. Let $f : Z \to Z : f(x) = 2x$. Find $g : Z \to Z : g \circ f = I_Z$.
10. Let $f : N \to N : f(x) = 2x$, $g : N \to N : g(y) = 3y + 4$ and $h : N \to N : h(z) = \sin z$.
Show that $h \circ (g \circ f) = (h \circ g) \circ f$.

11. If f be a greatest integer function and g be an absolute value function, find the value of $(f \circ g)\left(\frac{-3}{2}\right) + (g \circ f)\left(\frac{4}{3}\right)$. **[CBSE 2007]**

12. Let $f : R \to R : f(x) = x^2 + 2$ and $g : R \to R : g(x) = \frac{x}{x-1}$, $x \neq 1$. Find $f \circ g$ and $g \circ f$ and hence find $(f \circ g)(2)$ and $(g \circ f)(-3)$. **[CBSE 2014]**

ANSWERS (EXERCISE 2B)

1. (i) $(g \circ f) = \{(1, 4), (2, 3), (3, 2), (4, 1)\}$
(ii) $(f \circ g) = \{(1, 3), (2, 4), (3, 1), (4, 2)\}$
(iii) $(f \circ f) = \{(1, 2), (2, 4), (3, 3), (4, 1)\}$

2. (i) $(g \circ f) = \{(3, 3), (9, 3), (12, 9)\}$
(ii) $(f \circ g) = \{(1, 1), (3, 1), (4, 3), (5, 3)\}$

4. (i) $(g \circ f)(x) = (4x^2 + 4x - 1)$ (ii) $(f \circ g)(x) = (2x^2 - 3)$
(iii) $(f \circ f)(x) = (4x + 3)$ (iv) $(g \circ g)(x) = (x^4 - 4x^2 + 2)$

5. (i) $(g \circ f)(x) = (2x^2 + 6x - 1)$ (ii) $(f \circ g)(x) = (4x^2 - 6x + 1)$
(iii) $(g \circ g)(x) = (4x - 9)$

7. $[h \circ (g \circ f)](x) = \log(\tan x^2)$ **11.** 2

12. $(f \circ g)(x) = \frac{x^2}{(x-1)^2} + 2$, $(g \circ f)(x) = \frac{x^2+2}{x^2+1}$;
$(f \circ g)(2) = 6$, $(g \circ f)(-3) = \frac{11}{10}$.

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 2B)

9. Let $x \in Z$. Then,

$$(g \circ f) = I_Z \Rightarrow (g \circ f)(x) = I_Z(x)$$
$$\Rightarrow g[f(x)] = x$$
$$\Rightarrow g(2x) = x$$
$$\Rightarrow g(y) = \frac{1}{2}y \quad [\text{where } 2x = y].$$

Thus, $g : Z \to Z : g(y) = \frac{1}{2}y$.

11. $(f \circ g)\left(\frac{-3}{2}\right) = f\left\{g\left(\frac{-3}{2}\right)\right\} = f\left(\left|\frac{-3}{2}\right|\right) = f\left(\frac{3}{2}\right) = \left[\frac{3}{2}\right] = 1.$

$(g \circ f)\left(\frac{4}{3}\right) = g\left\{f\left(\frac{4}{3}\right)\right\} = g\left[\frac{4}{3}\right] = g(1) = |1| = 1.$

Required sum $= (1 + 1) = 2$.

12. $(f \circ g)(x) = f\{g(x)\} = f\left(\dfrac{x}{x-1}\right) = \dfrac{x^2}{(x-1)^2} + 2.$

$$(g \circ f)(x) = g\{f(x)\} = g(x^2+2) = \frac{x^2+2}{x^2+2-1} = \frac{x^2+2}{x^2+1}.$$

Invertible Function

Let $f : A \to B$. If there exists a function $g : B \to A$ such that $g \circ f = I_A$ and $f \circ g = I_B$ then f is called an invertible function and g is called the inverse of f. We write, $f^{-1} = g$.

REMARK Clearly, $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Example Let $f : R \to R : f(x) = 2x + 3$.

Let $y = f(x)$. Then,

$$y = f(x) \Rightarrow y = 2x + 3$$
$$\Rightarrow x = \frac{1}{2}(y-3)$$
$$\Rightarrow f^{-1}(y) = \frac{1}{2}(y-3) \; [\because \; y = f(x) \Rightarrow x = f^{-1}(y)].$$

Thus, we define:

$$f^{-1} : R \to R : f^{-1}(y) = \frac{1}{2}(y-3).$$

THEOREM 1 *If $f : A \to B$ is one-one onto then prove that f is an invertible function.*

PROOF Let $y \in B$. Then, f being one-one onto, there exists a unique $x \in A$ such that $f(x) = y$.

We define $g : B \to A : g(y) = x$. Then,

$$\begin{aligned}(g \circ f)(x) &= g[f(x)]\\ &= g(y) \quad [\because \; f(x) = y]\\ &= x \quad [\because \; g(y) = x]\\ &= I_A(x).\end{aligned}$$

$\therefore \; (g \circ f) = I_A$.

$$\begin{aligned}(f \circ g)(y) &= f[g(y)]\\ &= f(x) \quad [\because \; g(y) = x]\\ &= y \quad [\because \; f(x) = y]\\ &= I_B(y).\end{aligned}$$

$\therefore \; (f \circ g) = I_B$.

Hence, f is invertible and $f^{-1} = g$.

THEOREM 2 *If $f : A \to B$ is an invertible function, then prove that f is one-one onto.*

PROOF Let $f : A \to B$ be an invertible function. Then, there exists a function $g : B \to A$ such that

$$g \circ f = I_A \quad \text{and} \quad f \circ g = I_B.$$

Now, $f(x_1) = f(x_2)$

$\Rightarrow g\{f(x_1)\} = g\{f(x_2)\}$

$\Rightarrow (g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow I_A(x_1) = I_A(x_2) \quad [\because g \circ f = I_A]$

$\Rightarrow x_1 = x_2.$

$\therefore$ f is one-one.

Let $y \in B$. Then, $g(y) \in A$. Let $g(y) = x$. Then,

$g(y) = x$

$\Rightarrow f\{g(y)\} = f(x)$

$\Rightarrow (f \circ g)(y) = f(x)$

$\Rightarrow I_B(y) = f(x) \quad [\because f \circ g = I_B]$

$\Rightarrow y = f(x).$

Thus, for each $y \in B$ there exists $x \in A$ such that $y = f(x)$.

$\therefore$ f is onto.

Hence, f is one-one onto.

REMARK f is invertible $\Leftrightarrow$ f is one-one onto.

SOLVED EXAMPLES

EXAMPLE 1 *Let $f : R \to R : f(x) = 4x + 3$ for all $x \in R$. Show that f is invertible and find f^{-1}.*

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow 4x_1 + 3 = 4x_2 + 3$$
$$\Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Again, $y = 4x + 3 \Rightarrow x = \dfrac{(y-3)}{4}$.

Now, if $y \in R$ (codomain of f) then there exists $x = \dfrac{(y-3)}{4} \in R$

such that $f(x) = f\left(\dfrac{y-3}{4}\right) = \left\{4 \cdot \dfrac{(y-3)}{4} + 3\right\} = y.$

$\therefore$ f is onto.

Thus, f is one-one onto and therefore invertible.

Now, $y = f(x) \Rightarrow y = 4x + 3$

$$\Rightarrow x = \frac{(y-3)}{4}$$

$$\Rightarrow \quad f^{-1}(y) = \frac{(y-3)}{4} \qquad [\because \quad f(x) = y \;\Rightarrow\; x = f^{-1}(y)].$$

Thus, we define:

$$f^{-1} : R \to R : f^{-1}(y) = \frac{(y-3)}{4} \text{ for all } y \in R.$$

EXAMPLE 2 *Let R_+ be the set of all positive real numbers. Let $f : R_+ \to [4, \infty[: f(x) = x^2 + 4$. Show that f is invertible and find f^{-1}.*

SOLUTION We have

$$\begin{aligned} f(x_1) = f(x_2) &\Rightarrow x_1^2 + 4 = x_2^2 + 4 \\ &\Rightarrow x_1^2 = x_2^2 \\ &\Rightarrow x_1^2 - x_2^2 = 0 \\ &\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0 \\ &\Rightarrow x_1 - x_2 = 0 \qquad [\because \quad (x_1 + x_2) \neq 0] \\ &\Rightarrow x_1 = x_2. \end{aligned}$$

$\therefore$ f is one-one.

Now, $y = x^2 + 4 \Rightarrow x = \sqrt{y-4}$.

For each $y \in [4, \infty[$ there exists $x = \sqrt{y-4}$ in R_+ such that $f(x) = f(\sqrt{y-4}) = (\sqrt{y-4})^2 + 4 = (y-4) + 4 = y$.

$\therefore$ f is onto.

Thus, f is one-one onto and therefore invertible.

$$\begin{aligned} \text{Now, } y = f(x) &\Rightarrow y = x^2 + 4 \\ &\Rightarrow x = \sqrt{y-4} \\ &\Rightarrow f^{-1}(y) = \sqrt{y-4}. \end{aligned}$$

$\therefore$ $f^{-1} : [4, \infty[\to R_+ : f^{-1}(y) = \sqrt{y-4}$.

EXAMPLE 3 *Let R^+ be the set of all positive real numbers. Let $f : R^+ \to R^+ : f(x) = e^x$ for all $x \in R^+$. Show that f is invertible and hence find f^{-1}.*

SOLUTION f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2.$$

Now, for each $y \in R^+$, there exists a positive real number, namely $\log y$ such that

$$f(\log y) = e^{\log y} = y.$$

$\therefore$ f is onto.

Thus, f is one-one onto and hence invertible.

We define:

$$f^{-1}: R^+ \to R^+ : f^{-1}(y) = \log y \text{ for all } y \in R^+.$$

EXAMPLE 4 *Let* $A = \left\{x : x \in R,\ \frac{-\pi}{2} \le x \le \frac{\pi}{2}\right\}$ *and* $B = \{y : y \in R, -1 \le y \le 1\}$. *Show that the function* $f : A \to B : f(x) = \sin x$ *is invertible and hence find* f^{-1}.

SOLUTION Here, $A = \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ and $B = [-1, 1]$.

Also, $f : A \to B : f(x) = \sin x$.

f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow \sin x_1 = \sin x_2$$
$$\Rightarrow x_1 = x_2 \quad \left\{\because\ x_1, x_2 \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]\right\}.$$

$\therefore$ f is one-one.

Also, range $(f) = [-1, 1] = B$. So, f is onto.

Thus, f is one-one onto and hence invertible.

Now, $y = f(x) \Rightarrow y = \sin x$

$$\Rightarrow x = \sin^{-1} y$$
$$\Rightarrow f^{-1}(y) = \sin^{-1} y.$$

Thus, we define:

$$f^{-1} : [-1, 1] \to \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] : f^{-1}(y) = \sin^{-1} y.$$

EXAMPLE 5 *Let* $f : N \to Y : f(x) = x^2$, *where* $Y = \text{range}(f)$. *Show that f is invertible and find* f^{-1}.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2$$
$$\Rightarrow x_1^2 - x_2^2 = 0$$
$$\Rightarrow (x_1 - x_2)(x_1 + x_2) = 0$$
$$\Rightarrow x_1 - x_2 = 0 \quad [\because\ x_1 + x_2 \ne 0]$$
$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Since range $(f) = Y$, so f is onto.

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, there exists $x \in N$ such that $f(x) = y$.

Now, $y = f(x) \Rightarrow y = x^2$

$\Rightarrow x = \sqrt{y}$

$\Rightarrow f^{-1}(y) = \sqrt{y}$ $[\because f(x) = y \Rightarrow x = f^{-1}(y)]$.

Thus, we define

$$f^{-1} : Y \to N : f^{-1}(y) = \sqrt{y}.$$

EXAMPLE 6 *Let* $f : [-1, 1] \to Y : f(x) = \frac{x}{(x+2)}$, $x \neq -2$ *and* $Y = range(f)$. *Show that f is invertible and find* f^{-1}.

SOLUTION We have

$$f(x_1) = f(x_2) \Rightarrow \frac{x_1}{x_1 + 2} = \frac{x_2}{x_2 + 2}$$

$$\Rightarrow x_1x_2 + 2x_1 = x_1x_2 + 2x_2$$

$$\Rightarrow 2(x_1 - x_2) = 0$$

$$\Rightarrow x_1 - x_2 = 0$$

$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

Since range $(f) = Y$, so f is onto.

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, there exists $x \in [-1, 1]$ such that $f(x) = y$.

Now, $y = f(x) \Rightarrow y = \frac{x}{(x+2)}$

$$\Rightarrow x = \frac{2y}{(1-y)}$$

$$\Rightarrow f^{-1}(y) = \frac{2y}{(1-y)}.$$

Thus, we define:

$$f^{-1} : [-1, 1] \to Y : f^{-1}(y) = \frac{2y}{(1-y)}, \quad y \neq 1.$$

EXAMPLE 7 *Let* $f : N \to Y : f(x) = 4x^2 + 12x + 15$ *and* $Y = range(f)$. *Show that f is invertible and find* f^{-1}. **[CBSE 2013C]**

SOLUTION f is one-one, since

$$f(x_1) = f(x_2) \Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$$

$$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0$$

$$\Rightarrow (x_1^2 - x_2^2) + 3(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 3) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \quad [\because x_1 + x_2 + 3 \neq 0]$$

$$\Rightarrow x_1 = x_2.$$

Also, range $(f) = Y$. So, f is onto.

Thus, f is one-one onto and therefore invertible.

Let $y \in Y$. Then, f being onto, there exists x such that $y = f(x)$.

$$\text{Now, } y = f(x) \Rightarrow y = 4x^2 + 12x + 15$$
$$\Rightarrow y = (2x + 3)^2 + 6$$
$$\Rightarrow (2x + 3) = \sqrt{y - 6}$$
$$\Rightarrow x = \frac{1}{2}(\sqrt{y - 6} - 3)$$
$$\Rightarrow f^{-1}(y) = \frac{1}{2}(\sqrt{y - 6} - 3).$$

Thus, we define:

$$f^{-1} : Y \to N : f^{-1}(y) = \frac{1}{2}(\sqrt{y - 6} - 3).$$

EXAMPLE 8 *Let $f : R \to R : f(x) = 10x + 7$. Find the function $g : R \to R$ such that $g \circ f = f \circ g = I_R$.* **[CBSE 2011]**

SOLUTION Clearly, $g = f^{-1}$ … (i)

$$\text{Now, } f(x_1) = f(x_2) \Rightarrow 10x_1 + 7 = 10x_2 + 7$$
$$\Rightarrow 10x_1 = 10x_2$$
$$\Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

$$\text{Now, } y = f(x) \Rightarrow y = 10x + 7$$
$$\Rightarrow x = \frac{(y-7)}{10}.$$

Clearly, for each $y \in R$ (codomain of f) there exists $x \in R$ such that

$$f(x) = f\left(\frac{y-7}{10}\right) = \left\{10 \cdot \left(\frac{y-7}{10}\right) + 7\right\} = y.$$

$\therefore$ f is onto.

Thus, f is one-one onto and therefore, f^{-1} exists.

We define: $f^{-1} : R \to R : f^{-1}(y) = \dfrac{y-7}{10}$.

Hence, $g : R \to R : g(y) = \dfrac{y-7}{10}$ [using (i)].

EXAMPLE 9 *Let* $f : W \to W : f(n) = \begin{cases} (n-1), & \textit{when } n \textit{ is odd} \\ (n+1), & \textit{when } n \textit{ is even.} \end{cases}$

Show that f is invertible. Find f^{-1}. **[CBSE 2012]**

SOLUTION Let $f(n_1) = f(n_2)$.

Case 1 *When n_1 is odd and n_2 is even*

In this case, $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 + 1$
$\Rightarrow n_1 - n_2 = 2.$

If n_1 is odd and n_2 is even, then $(n_1 - n_2) \neq 2$.

Thus, we arrive at a contradiction.

So, in this case, $f(n_1) \neq f(n_2)$.

Similarly, when n_1 is even and n_2 is odd, then $f(n_1) \neq f(n_2)$.

Case 2 *When n_1 and n_2 are both odd*

In this case, $f(n_1) = f(n_2) \Rightarrow n_1 - 1 = n_2 - 1$
$\Rightarrow n_1 = n_2.$

Case 3 *When n_1 and n_2 are both even*

In this case, $f(n_1) = f(n_2) \Rightarrow n_1 + 1 = n_2 + 1$
$\Rightarrow n_1 = n_2.$

Thus, from all the cases, we get $f(n_1) = f(n_2) \Rightarrow n_1 = n_2$.

$\therefore$ f is one-one.

Now, we show that f is onto.

Let $n \in W$.

Case 1 *When n is odd*

In this case, $(n-1)$ is even

and $f(n-1) = (n-1) + 1 = n.$ … (i)

Case 2 *When n is even*

In this case, $(n+1)$ is odd

and $f(n+1) = (n+1) - 1 = n.$ … (ii)

Thus, each $n \in W$ has its pre-image in W.

$\therefore$ f is onto.

Thus, f is one-one onto and hence invertible.

Clearly, we have

$$f^{-1}(n) = \begin{cases} (n-1), & \text{when } n \text{ is odd} \\ (n+1), & \text{when } n \text{ is even} \end{cases} \quad \text{[using (i) and (ii)].}$$

EXAMPLE 10 *Let $A = \{1, 2, 3\}$ and let $f : A \to A$, defined by*
$f = \{(1, 2), (2, 3), (3, 1)\}$.
Find f^{-1}, if it exists.

SOLUTION We have $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$.

Dom $(f) = \{1, 2, 3\} = A$ and range $(f) = \{1, 2, 3\} = A$.

Clearly, different elements in A have different images.

$\therefore$ f is one-one.

Range $(f) = A \Rightarrow f$ is onto.

Thus, f is one-one onto and therefore invertible.

Now, $f(1) = 2$, $f(2) = 3$ and $f(3) = 1$

$\Rightarrow f^{-1}(2) = 1$, $f^{-1}(3) = 2$ and $f^{-1}(1) = 3$.

Hence, $f^{-1} = \{(2, 1), (3, 2), (1, 3)\}$.

Some Results on Invertible Functions

THEOREM 1 *Prove that an invertible function has a unique inverse.*

PROOF Let $f : A \to B$, which is one-one onto and therefore, invertible.
If possible, let it have two inverses, say g and h.

Then, $(f \text{ o } g) = I_B$ and $(f \text{ o } h) = I_B$

$\Rightarrow (f \text{ o } g)(y) = (f \text{ o } h)(y)$ [each $= I_B(y)$]

$\Rightarrow f\{g(y)\} = f\{h(y)\}$ for all $y \in B$

$\Rightarrow g(y) = h(y)$ for all $y \in B$ [$\because$ f is one-one].

$\therefore$ $g = h$.

Hence, f has a unique inverse.

THEOREM 2 *Let f be an invertible function. Then, prove that*

$$(f^{-1})^{-1} = f.$$

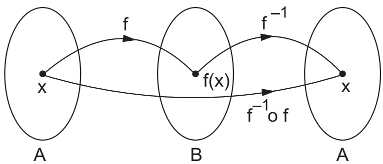
PROOF Let $f : A \to B$, which is invertible.
In order to prove that $(f^{-1})^{-1} = f$, it is sufficient to show that

$$f^{-1} \text{ o } f = I_A \text{ and } f \text{ o } f^{-1} = I_B.$$

Clearly $f : A \to B$ is one-one onto.

$\therefore$ $f^{-1} : B \to A$ is one-one onto.

Let $x \in A$ and let $f(x) = y$. Then, $f^{-1}(y) = x$.



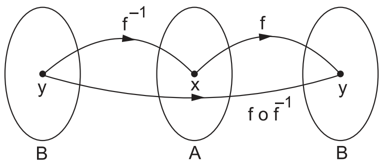
$\therefore$ $(f^{-1} \text{ o } f)(x) = f^{-1}\{f(x)\}$

$= f^{-1}(y)$ [$\because$ $f(x) = y$]

$= x$

$= I_A(x)$.

$\therefore$ $f^{-1} \text{ o } f = I_A$.



Again, let $y \in B$.

Then, f being onto, there exists $x \in A$ such that $f(x) = y$
and therefore, $f^{-1}(y) = x$.

$\therefore \quad (f \circ f^{-1})(y) = f\{f^{-1}(y)\}$
$= f(x) \quad [\because f^{-1}(y) = x]$
$= y$
$= I_B(y).$

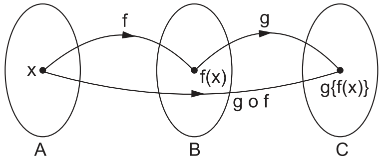
$\therefore \quad f \circ f^{-1} = I_B.$

Thus, $f^{-1} \circ f = I_A$ and $f \circ f^{-1} = I_B$.

Hence, $(f^{-1})^{-1} = f$.

THEOREM 3 *Let $f: A \to B$ and $g: B \to C$ be one-one onto functions. Prove that $(g \circ f): A \to C$ which is one-one onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

PROOF Let $f: A \to B$ be one-one onto and $g: B \to C$ be one-one onto.



We first show that $g \circ f$ is one-one onto.

$(g \circ f)$ is one-one, since

$(g \circ f)(x_1) = (g \circ f)(x_2)$

$\Rightarrow \quad g\{f(x_1)\} = g\{f(x_2)\}$

$\Rightarrow \quad f(x_1) = f(x_2) \quad [\because g$ is one-one$]$

$\Rightarrow \quad x_1 = x_2 \quad [\because f$ is one-one$]$.

Let $z \in C$. Then, g being onto, there exists $y \in B$ such that $g(y) = z$.

Now, f being onto, there exists $x \in A$ such that $f(x) = y$.

$\therefore \quad z = g(y)$
$= g\{f(x)\} \quad [\because y = f(x)]$
$= (g \circ f)(x).$

Thus, for each $z \in C$, there exists $x \in A$ such that $(g \circ f)(x) = z$.

$\therefore \quad (g \circ f)$ is onto.

Thus, $(g \circ f)$ is one-one onto.

Now, $f(x) = y \Rightarrow f^{-1}(y) = x$.

And, $g(y) = z \Rightarrow g^{-1}(z) = y$.

Also, $(g \circ f)(x) = z \Rightarrow (g \circ f)^{-1}(z) = x$.

$\therefore \quad (f^{-1} \circ g^{-1})(z) = f^{-1}\{g^{-1}(z)\}$
$= f^{-1}(y) \quad [\because g^{-1}(z) = y]$
$= x \quad [\because f^{-1}(y) = x]$
$= (g \circ f)^{-1}(z).$

Hence, $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$.

EXERCISE 2C

Very-Short-Answer Questions

1. Prove that the function $f : R \to R : f(x) = 2x$ is one-one and onto.
2. Prove that the function $f : N \to N : f(x) = 3x$ is one-one and into.
3. Show that the function $f : R \to R : f(x) = x^2$ is neither one-one nor onto.
4. Show that the function $f : N \to N : f(x) = x^2$ is one-one and into.
5. Show that the function $f : R \to R : f(x) = x^4$ is neither one-one nor onto.
6. Show that the function $f : Z \to Z : f(x) = x^3$ is one-one and into.
7. Let R_0 be the set of all nonzero real numbers. Then, show that the function $f : R_0 \to R_0 : f(x) = \frac{1}{x}$ is one-one and onto.
8. Show that the function $f : R \to R : f(x) = 1 + x^2$ is many-one into.
9. Let $f : R \to R : f(x) = \frac{2x-7}{4}$ be an invertible function. Find f^{-1}.

[CBSE 2008C]

10. Let $f : R \to R : f(x) = 10x + 3$. Find f^{-1}.
11. $f : R \to R : f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is rational.} \end{cases}$

    Show that f is many-one and into.
12. Let $f(x) = x + 7$ and $g(x) = x - 7$, $x \in R$. Find $(f \circ g)(7)$. **[CBSE 2008]**
13. Let $f : R \to R$ and $g : R \to R$ defined by $f(x) = x^2$ and $g(x) = (x+1)$. Show that $g \circ f \neq f \circ g$.
14. Let $f : R \to R : f(x) = (3 - x^3)^{1/3}$. Find $f \circ f$. **[CBSE 2010]**
15. Let $f : R \to R : f(x) = 3x + 2$, find $f\{f(x)\}$. **[CBSE 2010C]**
16. Let $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(1, 3), (2, 3), (5, 1)\}$. Write down $g \circ f$. **[CBSE 2014C]**
17. Let $A = \{1, 2, 3, 4\}$ and $f = \{(1, 4), (2,1)\ (3, 3), (4, 2)\}$. Write down $(f \circ f)$.
18. Let $f(x) = 8x^3$ and $g(x) = x^{1/3}$. Find $g \circ f$ and $f \circ g$.
19. Let $f : R \to R : f(x) = 10x + 7$. Find the function $g : R \to R : g \circ f = f \circ g = I_g$. **[CBSE 2011]**
20. Let $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$ and let $f = \{(1, 4), (2, 5), (3, 6)\}$ be a function from A to B. State whether f is one-one. **[CBSE 2011]**

ANSWERS (EXERCISE 2C)

9. $f^{-1}(y) = \frac{1}{2}(4y+7)$ for all $y \in R$ **10.** $f^{-1}(y) = \frac{1}{10}(y-3)$ for all $y \in R$

12. 7 **14.** $(f \circ f)(x) = x$ **15.** $f\{f(x)\} = (9x+8)$

16. $(g \circ f) = \{(1, 3), (3, 1), (4, 3)\}$ **17.** $f \circ f = \{(1, 2), (2, 4), (3, 3), (4, 1)\}$

18. $(g \circ f)(x) = 2x$ and $(f \circ g)(x) = 8x$ **19.** $g(x) = \frac{1}{10}(x-7)$ for all $x \in R$

20. Yes

HINTS TO THE GIVEN QUESTIONS (EXERCISE 2C)

1. (i) $f(x_1) = f(x_2) \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$. So, f is one-one.

   (ii) Let $y = 2x$. Then, $x = \frac{1}{2}y$.

   Thus for each y in codomain R, there exists $\frac{1}{2}y$ such that

   $$f\left(\frac{1}{2}y\right) = \left(2 \times \frac{1}{2}y\right) = y.$$

   $\therefore$ f is onto.

2. (i) $f(x_1) = f(x_2) \Rightarrow 3x_1 = 3x_2 \Rightarrow x_1 = x_2$. So, f is one-one.

   (ii) If we consider 2 in codomain N, there is no natural number whose image is 2. So, f is into.

3. (i) Clearly, $f(1) = 1^2 = 1$ and $f(-1) = (-1)^2 = 1$.

   So, f is many-one.

   (ii) If we consider –1 in the codomain R then there is no element in R whose square is –1.

   $\therefore$ $-1 \in R$ has no pre-image in R. So, f is many-one into.

4. (i) $f(x_1) = f(x_2) \Rightarrow x_1^2 = x_2^2 \Rightarrow x_1 = x_2$ $[\because x_1, x_2 \in N]$.

   $\therefore$ f is one-one.

   (ii) If we consider 2 in the codomain N, then $\sqrt{2} \in N$ and $f(\sqrt{2}) = (\sqrt{2})^2 = 2$.

   So, f is into.

5. (i) $f(1) = 1^4 = 1$ and $f(-1) = (-1)^4 = 1$.

   So, f is many-one.

   (ii) If we consider –1 in the codomain R then there exists no $x \in R$ such that $f(x) = x^4 = -1$. So, f is into.

6. (i) Let $x_1, x_2 \in Z$ and $x_1 \neq x_2$. Then, $x_1 \neq x_2 \Rightarrow x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_2)$.

   (ii) Let $2 \in Z$. Then, there exists no $x \in Z$ such that $x^3 = 2$.

   Thus, $2 \in Z$ has no pre-image in Z. So, f is into.

7. (i) $f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2} \Rightarrow x_1 = x_2$. So, f is one-one.

(ii) $y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$.

Thus, for each y in codomain R_0, there exists $\frac{1}{y}$ in domain R_0 such that $f\left(\frac{1}{y}\right) = \frac{1}{\left(\frac{1}{y}\right)} = y$. So, f is onto.

8. (i) $f(-1) = 1 + (-1)^2 = 2$ and $f(1) = (1 + 1^2) = 2$.

So, f is many-one.

(ii) $y = (1 + x^2) \Rightarrow x = \sqrt{y - 1}$.

So, when $y < 1$, then $\sqrt{y-1}$ is imaginary.

In particular, $0 \in R$ has no pre-image in R.

$\therefore$ f is into.

9. $y = \frac{2x-7}{4} \Rightarrow x = \frac{1}{2}(4y + 7)$

$$\Rightarrow f^{-1}(y) = \frac{4y+7}{2} \text{ for all } y \in R.$$

10. $y = 10x + 3 \Rightarrow x = \frac{y-3}{10}$

$$\Rightarrow f^{-1}(y) = \frac{(y-3)}{10}.$$

11. (i) Since all rationals have the same image, namely 1, so f is many-one.

(ii) Range $(f) = \{-1\ 1\} \subset R$. So, f is into.

12. $(f \circ g)(7) = f\{g(7)\} = f(7-7) = f(0) = (0+7) = 7$.

13. $(g \circ f)(x) = g\{f(x)\} = g(x^2) = (x^2 + 1)$.

$(f \circ g)(x) = f\{g(x)\} = f(x+1) = (x+1)^2$.

Hence, $g \circ f \neq f \circ g$.

14. $(f \circ f)(x) = f\{f(x)\} = f\{(3 - x^3)^{1/3}\} = f(y)$, where $y = (3 - x^3)^{1/3}$

$$= (3 - y^3)^{1/3} = \{3 - (3 - x^3)\}^{1/3} = (x^3)^{1/3} = x.$$

$\therefore$ $(f \circ f)(x) = x$.

15. $f\{f(x)\} = f(3x+2) = 3(3x+2) + 2 = (9x + 8)$.

16. Dom $(g \circ f)$ = Dom $(f) = \{1, 3, 4\}$.

$(g \circ f)(1) = g\{f(1)\} = g(2) = 3$.

$(g \circ f)(3) = g\{f(3)\} = g(5) = 1$.

$(g \circ f)(4) = g\{f(4)\} = g(1) = 3$.

$\therefore$ $g \circ f = \{(1, 3), (3, 1), (4, 3)\}$.

17. $(f \circ f)(1) = f\{f(1)\} = f(4) = 2$.

$(f \circ f)(2) = f\{f(2)\} = f(1) = 4$.

$(f \circ f)(3) = f\{f(3)\} = f(3) = 3.$
$(f \circ f)(4) = f\{f(4)\} = f(2) = 1.$
$\therefore \quad f \circ f = \{(1, 2), (2, 4), (3, 3), (4, 1)\}.$

18. $(g \circ f)(x) = g[f(x)] = g(8x^3) = g(y)$, where $y = 8x^3$
$= y^{1/3} = (8x^3)^{1/3} = 2x.$
$(f \circ g)(x) = f[g(x)] = f(x^{1/3}) = f(y)$, where $y = x^{1/3}$
$= 8y^3 = 8(x^{1/3})^3 = 8x^{\left(\frac{1}{3} \times 3\right)} = 8x.$

19. $g \circ f = I_R \Rightarrow (g \circ f)(x) = I_R(x) = x$
$\Rightarrow g\{f(x)\} = x$
$\Rightarrow g\{10x + 7\} = x.$
Put $10x + 7 = y$. Then, $x = \frac{(y-7)}{10}$.
$\therefore \quad g(y) = \frac{(y-7)}{10}$.
Hence, $g : R \to R : g(x) = \frac{1}{10}(x-7)$ for all $x \in R$.

20. $f(1) = 4, f(2) = 5$ and $f(3) = 6$.
Thus, different elements in A have different images in B.
Hence, f is one-one.

EXERCISE 2D

1. Let $A = \{2, 3, 4, 5\}$ and $B = \{7, 9, 11, 13\}$, and let $f = \{(2, 7), (3, 9), (4, 11), (5, 13)\}$.
Show that f is invertible and find f^{-1}.
2. Show that the function $f : R \to R : f(x) = 2x + 3$ is invertible and find f^{-1}.
3. Let $f : Q \to Q : f(x) = 3x - 4$. Show that f is invertible and find f^{-1}.
4. Let $f : R \to R : f(x) = \frac{1}{2}(3x + 1)$. Show that f is invertible and find f^{-1}.
5. If $f(x) = \frac{(4x + 3)}{(6x - 4)}$, $x \neq \frac{2}{3}$, show that $(f \circ f)(x) = x$ for all $x \neq \frac{2}{3}$.
Hence, find f^{-1}.
6. Show that the function f on $A = R - \left\{\frac{2}{3}\right\}$, defined as $f(x) = \frac{4x + 3}{6x - 4}$ is one-one and onto. Hence, find f^{-1}. **[CBSE 2013]**
7. Show that the function f on $A = R - \left\{\frac{-4}{3}\right\}$ into itself, defined by $f(x) = \frac{4x}{(3x + 4)}$ is one-one and onto. Hence, find f^{-1}.

8. Let R_+ be the set of all positive real numbers. Show that the function $f : R_+ \to [-5, \infty[: f(x) = (9x^2 + 6x - 5)$ is invertible. Find f^{-1}.

9. Let $f : N \to R : f(x) = 4x^2 + 12x + 15$. Show that $f : N \to$ range (f) is invertible. Find f^{-1}. **[CBSE 2010, '13C]**

10. Let $A = R - \{2\}$ and $B = R - \{1\}$. If $f : A \to B : f(x) = \dfrac{x-1}{x-2}$, show that f is one-one and onto. Hence, find f^{-1}. **[CBSE 2013]**

11. Let f and g be two functions from R into R, defined by $f(x) = |x| + x$ and $g(x) = |x| - x$ for all $x \in R$. Find $f \circ g$ and $g \circ f$. **[CBSE 2014C]**

ANSWERS (EXERCISE 2D)

1. $f^{-1} = \{(7, 2), (9, 3), (11, 4), (13, 5)\}$ **2.** $f^{-1}(y) = \dfrac{1}{2}(y-3)$

3. $f^{-1}(y) = \dfrac{1}{3}(y+4)$ **4.** $f^{-1}(y) = \dfrac{(2y-1)}{3}$

5. $f^{-1}(y) = \dfrac{(4y+3)}{(6y-4)}$ **6.** $f^{-1}(y) = \dfrac{(4y+3)}{6y-4}$

7. $f^{-1}(y) = \dfrac{(4y)}{(4-3y)}$ **8.** $f^{-1}(y) = \dfrac{\sqrt{y+6}-1}{3}$

9. $f^{-1}(y) = \dfrac{\sqrt{y-6}-3}{2}$ **10.** $f^{-1}(y) = \dfrac{2y-1}{y-1}$

11. $(f \circ g)(x) = |x| - x$ and $(g \circ f)(x) = |x| - x$

HINTS TO SOME SELECTED QUESTIONS (EXERCISE 2D)

1. Clearly $f(2) = 7, f(3) = 9, f(4) = 11$ and $f(5) = 13$.
Thus, different elements in A have different images in B.
So, f is one-one.
Range $(f) = \{7, 9, 11, 13\} = B$. So, f is onto.
$f^{-1} = \{(7, 2), (9, 3), (11, 4), (13, 5)\}$.

2. $f(x_1) = f(x_2) \Rightarrow 2x_1 + 3 = 2x_2 + 3 \Rightarrow 2x_1 = 2x_2 \Rightarrow x_1 = x_2$.
$\therefore$ f is one-one.
If $y \in R$ then there exists $x = \dfrac{y-3}{2} \in R$ such that
$$f(x) = f\left(\frac{y-3}{2}\right) = \left\{2 \cdot \frac{(y-3)}{2} + 3\right\} = y.$$
$\therefore$ f is onto.

$$y = f(x) \Rightarrow y = 2x + 3$$
$$\Rightarrow x = \frac{1}{2}(y - 3) \Rightarrow f^{-1}(y) = \frac{1}{2}(y - 3).$$

5. $(f \circ f)(x) = f\{f(x)\} = f\left\{\frac{4x + 3}{6x - 4}\right\} = \dfrac{4\left(\dfrac{4x + 3}{6x - 4}\right) + 3}{6\left(\dfrac{4x + 3}{6x - 4}\right) - 4}$

$$= \frac{(16x + 12 + 18x - 12)}{(24x + 18 - 24x + 16)} = \frac{34x}{34} = x = I(x).$$

$\therefore \quad f \circ f = I \Rightarrow f^{-1} = f.$

6. $f(x_1) = f(x_2) \Rightarrow \dfrac{4x_1 + 3}{6x_1 - 4} = \dfrac{4x_2 + 3}{6x_2 - 4}$

$$\Rightarrow (4x_1 + 3)(6x_2 - 4) = (6x_1 - 4)(4x_2 + 3)$$
$$\Rightarrow 24x_1x_2 - 16x_1 + 18x_2 - 12 = 24x_1x_2 + 18x_1 - 16x_2 - 12$$
$$\Rightarrow 34x_1 = 34x_2 \Rightarrow x_1 = x_2.$$

$\therefore \quad f$ is one-one.

Let y be an arbitrary element of A. Then,

$$f(x) = y \Rightarrow \frac{4x + 3}{6x - 4} = y$$
$$\Rightarrow (6x - 4)y = 4x + 3 \Rightarrow 6xy - 4y = 4x + 3$$
$$\Rightarrow 6xy - 4x = 4y + 3 \Rightarrow x(6y - 4) = (4y + 3)$$
$$\Rightarrow x = \frac{4y + 3}{6y - 4}.$$

Thus, for each $y \in A$, there exists $x = \dfrac{4y + 3}{6y - 4}$ such that

$$f(x) = f\left(\frac{4y + 3}{6y - 4}\right) = \frac{4 \cdot \left(\dfrac{4y + 3}{6y - 4}\right) + 3}{6 \cdot \left(\dfrac{4y + 3}{6y - 4}\right) - 4}$$

$$= \frac{16y + 12 + 18y - 12}{24y + 18 - 24y + 16} = \frac{34y}{34} = y.$$

$\therefore \quad f$ is onto.

Now, $f(x) = y \Rightarrow x = f^{-1}(y)$.

$\therefore \quad f^{-1}(y) = \dfrac{4y + 3}{6y - 4}$, for all $y \in A$.

7. $f(x_1) = f(x_2) \Rightarrow \dfrac{4x_1}{3x_1 + 4} = \dfrac{4x_2}{3x_2 + 4} \Rightarrow \dfrac{x_1}{3x_1 + 4} = \dfrac{x_2}{3x_2 + 4}$

$$\Rightarrow x_1(3x_2 + 4) = x_2(3x_1 + 4)$$
$$\Rightarrow 3x_1x_2 + 4x_1 = 3x_1x_2 + 4x_2$$
$$\Rightarrow 4x_1 = 4x_2 \Rightarrow x_1 = x_2.$$

$\therefore \quad f$ is one-one.

Let y be an arbitrary element of A. Then,

$$f(x) = y \Rightarrow \frac{4x}{3x+4} = y$$

$$\Rightarrow 3xy + 4y = 4x \Rightarrow 4x - 3xy = 4y$$

$$\Rightarrow x(4-3y) = 4y \Rightarrow x = \frac{4y}{(4-3y)}.$$

Thus, for each $y \in A$, there exists an $x \in A$ such that

$$f(x) = f\left(\frac{4y}{4-3y}\right) = \frac{4\cdot\left(\frac{4y}{4-3y}\right)}{3\cdot\left(\frac{4y}{4-3y}\right)+4}$$

$$= \frac{16y}{12y + 16 - 12y} = \frac{16y}{16} = y.$$

$\therefore$ f is onto.

Now, $f(x) = y \Rightarrow x = f^{-1}(y)$.

$$\therefore \quad f^{-1}(y) = \frac{4y}{4-3y}.$$

8. $y = (3x+1)^2 - 6 \Rightarrow x = \dfrac{\sqrt{y+6}-1}{3} \Rightarrow f^{-1}(y) = \dfrac{\sqrt{y+6}-1}{3}$.

9. $f(x_1) = f(x_2) \Rightarrow 4x_1^2 + 12x_1 + 15 = 4x_2^2 + 12x_2 + 15$

$$\Rightarrow 4(x_1^2 - x_2^2) + 12(x_1 - x_2) = 0 \Rightarrow (x_1^2 - x_2^2) + 3(x_1 - x_2) = 0$$

$$\Rightarrow (x_1 - x_2)(x_1 + x_2 + 3) = 0 \Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

Since $f : N \to \text{range}(f)$, so f is onto.

Now $4x^2 + 12x + 15 = y \Rightarrow (2x+3)^2 + 6 = y$

$$\Rightarrow (2x+3) = \sqrt{y-6} \Rightarrow x = \frac{\sqrt{y-6}-3}{2}.$$

$$\therefore \quad f^{-1}(y) = \frac{\sqrt{y-6}-3}{2}.$$

OBJECTIVE QUESTIONS

Mark (✓) against the correct answer in each of the following:

1. $f : N \to N : f(x) = 2x$ is
   (a) one-one and onto (b) one-one and into
   (c) many-one and onto (d) many-one and into
2. $f : N \to N : f(x) = x^2 + x + 1$ is
   (a) one-one and onto (b) one-one and into
   (c) many-one and onto (d) many-one and into

3. $f : R \to R : f(x) = x^2$ is

(a) one-one and onto
(b) one-one and into
(c) many-one and onto
(d) many-one and into

4. $f : R \to R : f(x) = x^3$ is

(a) one-one and onto
(b) one-one and into
(c) many-one and onto
(d) many-one and into

5. $f : R^+ \to R^+ : f(x) = e^x$ is

(a) many-one and into
(b) many-one and onto
(c) one-one and into
(d) one-one and onto

6. $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1] : f(x) = \sin x$ is

(a) one-one and into
(b) one-one and onto
(c) many-one and into
(d) many-one and onto

7. $f : R \to R : f(x) = \cos x$ is

(a) one-one and into
(b) one-one and onto
(c) many-one and into
(d) many-one and onto

8. $f : C \to R : f(z) = |z|$ is

(a) one-one and into
(b) one-one and onto
(c) many-one and into
(d) many-one and onto

9. Let $A = R - \{3\}$ and $B = R - \{1\}$. Then, $f : A \to B : f(x) = \frac{(x-2)}{(x-3)}$ is

(a) one-one and into
(b) one-one and onto
(c) many-one and into
(d) many-one and onto

10. Let $f : N \to N : f(n) = \begin{cases} \frac{1}{2}(n+1), \text{ when } n \text{ is odd} \\ \frac{n}{2}, \text{ when } n \text{ is even.} \end{cases}$

Then, f is

(a) one-one and into
(b) one-one and onto
(c) many-one and into
(d) many-one and onto

11. Let A and B be two non-empty sets and let $f : (A \times B) \to (B \times A) : f(a, b) = (b, a)$. Then, f is

(a) one-one and onto
(b) one-one and into
(c) many-one and onto
(d) many-one and into

12. Let $f : Q \to Q : f(x) = (2x + 3)$. Then, $f^{-1}(y) = ?$

(a) $(2y - 3)$
(b) $\frac{1}{(2y-3)}$
(c) $\frac{1}{2}(y - 3)$
(d) none of these

13. Let $f : R - \left\{\frac{-4}{3}\right\} \to R - \left\{\frac{4}{3}\right\} : f(x) = \frac{4x}{(3x+4)}$. Then, $f^{-1}(y) = ?$

(a) $\frac{4y}{(4-3y)}$ (b) $\frac{4y}{(4y+3)}$ (c) $\frac{4y}{(3y-4)}$ (d) none of these

14. Let $f : N \to X : f(x) = 4x^2 + 12x + 15$. Then, $f^{-1}(y) = ?$

(a) $\frac{1}{2}(\sqrt{y-4}+3)$ (b) $\frac{1}{2}(\sqrt{y-6}-3)$

(c) $\frac{1}{2}(\sqrt{y-4}+5)$ (d) none of these

15. If $f(x) = \frac{(4x+3)}{(6x-4)}, x \neq \frac{2}{3}$ then $(f \circ f)(x) = ?$

(a) x (b) $(2x-3)$ (c) $\frac{4x-6}{3x+4}$ (d) none of these

16. If $f(x) = (x^2 - 1)$ and $g(x) = (2x+3)$ then $(g \circ f)(x) = ?$

(a) $(2x^2+3)$ (b) $(3x^2+2)$ (c) $(2x^2+1)$ (d) none of these

17. If $f\left(x + \frac{1}{x}\right) = x^2 + \frac{1}{x^2}$ then $f(x) = ?$

(a) x^2 (b) (x^2-1) (c) (x^2-2) (d) none of these

18. If $f(x) = \frac{1}{(1-x)}$ then $(f \circ f \circ f)(x) = ?$

(a) $\frac{1}{(1-3x)}$ (b) $\frac{x}{(1+3x)}$ (c) x (d) none of these

19. If $f(x) = \sqrt[3]{3-x^3}$ then $(f \circ f)(x) = ?$

(a) $x^{1/3}$ (b) x (c) $(1-x^{1/3})$ (d) none of these

20. If $f(x) = x^2 - 3x + 2$ then $(f \circ f)(x) = ?$

(a) x^4 (b) $x^4 - 6x^3$ (c) $x^4 - 6x^3 + 10x^2$ (d) none of these

21. If $f(x) = 8x^3$ and $g(x) = x^{1/3}$ then $(g \circ f)(x) = ?$

(a) x (b) $2x$ (c) $\frac{x}{2}$ (d) $3x^2$

22. If $f(x) = x^2$, $g(x) = \tan x$ and $h(x) = \log x$ then $\{h \circ (g \circ f)\}\left(\sqrt{\frac{\pi}{4}}\right) = ?$

(a) 0 (b) 1 (c) $\frac{1}{x}$ (d) $\frac{1}{2}\log\frac{\pi}{4}$

23. If $f = \{(1, 2), (3, 5), (4, 1)\}$ and $g = \{(2, 3), (5, 1), (1, 3)\}$ then $(g \circ f) = ?$

(a) $\{(3, 1), (1, 3), (3, 4)\}$ (b) $\{(1, 3), (3, 1), (4, 3)\}$

(c) $\{(3, 4), (4, 3), (1, 3)\}$ (d) $\{(2, 5), (5, 2), (1, 5)\}$

24. Let $f(x) = \sqrt{9 - x^2}$. Then, dom $(f) = ?$

(a) $[-3, 3]$
(b) $(-\infty, -3]$
(c) $[3, \infty)$
(d) $(-\infty, -3] \cup (4, \infty)$

25. Let $f(x) \sqrt{\frac{x-1}{x-4}} \cdot$ Then, dom $(f) = ?$

(a) $[1, 4)$
(b) $[1, 4]$
(c) $(-\infty, 4]$
(d) $(-\infty, 1] \cup (4, \infty)$

26. Let $f(x) = e^{\sqrt{x^2-1}} \cdot \log(x-1)$. Then, dom $(f) = ?$

(a) $(-\infty, 1]$
(b) $[-1, \infty)$
(c) $(1, \infty)$
(d) $(-\infty, -1] \cup (1, \infty)$

27. Let $f(x) = \frac{x}{(x^2-1)} \cdot$ Then, dom $(f) = ?$

(a) R
(b) $R - \{1\}$
(c) $R - \{-1\}$
(d) $R - \{-1, 1\}$

28. Let $f(x) = \frac{\sin^{-1} x}{x} \cdot$ Then, dom $(f) = ?$

(a) $(-1, 1)$
(b) $[-1, 1]$
(c) $[-1, 1] - \{0\}$
(d) none of these

29. Let $f(x) = \cos^{-1} 2x$. Then, dom $(f) = ?$

(a) $[-1, 1]$
(b) $\left[\frac{-1}{2}, \frac{1}{2}\right]$
(c) $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$
(d) $\left[\frac{-\pi}{4,} \frac{\pi}{4}\right]$

30. Let $f(x) = \cos^{-1}(3x - 1)$. Then, dom $(f) = ?$

(a) $\left(0, \frac{2}{3}\right)$
(b) $\left[0, \frac{2}{3}\right]$
(c) $\left[\frac{-2}{3}, \frac{2}{3}\right]$
(d) none of these

31. Let $f(x) = \sqrt{\cos x}$. Then, dom $(f) = ?$

(a) $\left[0, \frac{\pi}{2}\right]$
(b) $\left[\frac{3\pi}{2}, 2\pi\right]$
(c) $\left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$
(d) none of these

32. Let $f(x) = \sqrt{\log(2x - x^2)}$. Then, dom $(f) = ?$

(a) $(0, 2)$
(b) $[1, 2]$
(c) $(-\infty, 1]$
(d) none of these

33. Let $f(x) = x^2$. Then, dom (f) and range (f) are respectively

(a) R and R
(b) R^+ and R^+
(c) R and R^+
(d) R and $R - \{0\}$

34. Let $f(x) = x^3$. Then, dom (f) and range (f) are respectively

(a) R and R
(b) R^+ and R^+
(c) R and R^+
(d) R^+ and R

35. Let $f(x) = \log(1-x) + \sqrt{x^2-1}$. Then, dom $(f) = ?$

(a) $(1, \infty)$ (b) $(-\infty, -1]$ (c) $[-1, 1)$ (d) $(0, 1)$

36. Let $f(x) = \frac{1}{(1-x^2)}$ · Then, range $(f) = ?$

(a) $(-\infty, 1]$ (b) $[1, \infty)$ (c) $[-1, 1]$ (d) none of these

37. Let $f(x) = \frac{x^2}{(1+x^2)}$ · Then, range $(f) = ?$

(a) $[1, \infty)$ (b) $[0, 1)$ (c) $[-1, 1]$ (d) $(0, 1]$

38. The range of $f(x) = x + \frac{1}{x}$ is

(a) $[-2, 2]$ (b) $[2, \infty)$ (c) $(-\infty, -2]$ (d) none of these

39. The range of $f(x) = a^x$, where $a > 0$ is

(a) $]-\infty, 0]$ (b) $]-\infty, 0)$ (c) $[0, \infty)$ (d) $(0, \infty)$

ANSWERS (OBJECTIVE QUESTIONS)

1. (b) **2.** (b) **3.** (d) **4.** (a) **5.** (d) **6.** (b) **7.** (c) **8** (c) **9.** (b) **10.** (d)
11. (a) **12.** (c) **13.** (a) **14.** (b) **15.** (a) **16.** (c) **17.** (c) **18.** (c) **19.** (b) **20.** (d)
21. (b) **22.** (a) **23.** (b) **24.** (a) **25.** (d) **26.** (c) **27.** (d) **28.** (c) **29.** (b) **30.** (b)
31. (c) **32.** (d) **33.** (c) **34.** (a) **35.** (b) **36.** (b) **37.** (b) **38.** (d) **39.** (d)

HINTS TO SOME SELECTED OBJECTIVE QUESTIONS

1. $2x = 3 \Rightarrow x = \frac{3}{2} \notin N$. So, f is into.

2. $f(x_1) = f(x_2) \Rightarrow x_1^2 + x_1 + 1 = x_2^2 + x_2 + 1$

$$\Rightarrow (x_1^2 - x_2^2) + (x_1 - x_2) = 0 \Rightarrow (x_1 - x_2)(x_1 + x_2 + 1) = 0$$

$$\Rightarrow x_1 - x_2 = 0 \Rightarrow x_1 = x_2.$$

$\therefore$ f is one-one.

$f(x) = 1 \Rightarrow x^2 + x + 1 = 1 \Rightarrow x(x+1) = 0 \Rightarrow x = 0$ or $x = -1$.

And, none of 0 and –1 is in N. So, f is into.

5. $f(x_1) = f(x_2) \Rightarrow e^{x_1} = e^{x_2} \Rightarrow x_1 = x_2$. So, f is one-one.

For each $x \in R^+$ $\exists \log x \in R^+$ s.t. $f(\log x) = x$.

So, f is onto.

7. $\cos(2\pi - \theta) = \cos\theta \Rightarrow f$ is many-one.

Range $(f) = [-1, 1] \subset R \Rightarrow f$ is into.

8. $i \neq -i$. But $f(i) = f(-i) = 1$. So, f is many-one.

$-1 \in R$ having no pre-image in C. So, f is into.

9. $f(x_1) = f(x_2) \Rightarrow \dfrac{(x_1 - 2)}{(x_1 - 3)} = \dfrac{(x_2 - 2)}{(x_3 - 3)} \Rightarrow x_1 = x_2$. So, f is one-one.

Let $\dfrac{x-2}{x-3} = y$. Then, $x = \dfrac{3y-2}{y-1} \cdot$ Clearly, $y \neq 1$ and $x \neq 3$.

$\therefore$ $f(x) = y$ and so f is onto.

10. $f(1) = f(2)$ shows that f is many-one.

If n is odd then $(2n - 1)$ is odd and $f(2n - 1) = n$.

If n is even then $2n$ is even and $f(2n) = n$.

$\therefore$ f is onto.

12. $y = 2x + 3 \Rightarrow x = \frac{1}{2}(y - 3) \Rightarrow f^{-1}(y) = \frac{1}{2}(y - 3)$.

13. $y = \dfrac{4x}{3x + 4} \Rightarrow x = \dfrac{4y}{(4 - 3y)} \Rightarrow f^{-1}(y) = \dfrac{4y}{(4 - 3y)} \cdot$

14. $y = 4x^2 + 12x + 15 = (2x + 3)^2 + 6 \Rightarrow x = \frac{1}{2}(\sqrt{y - 6} - 3)$

$\therefore$ $f^{-1}(y) = \frac{1}{2}(\sqrt{y - 6} - 3)$.

15. $f(x) = \dfrac{4x + 3}{6x - 4} = y$ (say).

$$\text{Then } f(y) = \frac{4y + 3}{6y - 4} = \frac{4\left(\dfrac{4x+3}{6x-4}\right) + 3}{6\left(\dfrac{4x+3}{6x-4}\right) - 4} = \frac{34x}{34} = x.$$

$\Rightarrow$ $f[f(x)] = x \Rightarrow (f \text{ o} f)(x) = x$.

16. $(g \text{ o} f)(x) = g[f(x)] = g(x^2 - 1)$

$= 2(x^2 - 1) + 3 = (2x^2 + 1)$.

17. Let $x + \dfrac{1}{x} = z$. Then,

$$f(z) = f\left(x + \frac{1}{x}\right) = \left(x^2 + \frac{1}{x^2}\right) = \left(x + \frac{1}{x}\right)^2 - 2 = (z^2 - 2).$$

$\Rightarrow$ $f(x) = (x^2 - 2)$.

18. $(f \text{ o} f)(x) = f\{f(x)\} = f\left(\dfrac{1}{1-x}\right) = \dfrac{1}{\left(1 - \dfrac{1}{1-x}\right)} = \dfrac{1-x}{-x} = \dfrac{x-1}{x}$

$\Rightarrow$ $\{f \text{ o}(f \text{ o} f)\}(x) = f\{(f \text{ o} f)(x)\} = f\left(\dfrac{x-1}{x}\right) = \dfrac{1}{1 - \dfrac{x-1}{x}} = x$.

19. $(f \text{ o} f)(x) = f\{f(x)\} - \{(3 - x^3)^{1/3}\} = f(y)$, where $y = (3 - x^3)^{1/3}$

$= (3 - y^3)^{1/3} = [3 - \{3 - x^3\}]^{1/3} = (x^3)^{1/3} = x$.

20. $(f \circ f)(x) = f\{f(x)\} = f(x^2 - 3x + 2) = f(y)$, where $y = (x^2 - 3x + 2)$

$$= y^2 - 3y + 2 = (x^2 - 3x + 2)^2 - 3(x^2 - 3x + 2) + 2$$

$$= (x^4 - 6x^3 + 10x^2 - 3x).$$

21. $(g \circ f)(x) = g[f(x)] = g(8x^3) = (8x^3)^{1/3} = 2x.$

22. $\{h \circ (g \circ f)\}(x) = (h \circ g)\{f(x)\} = (h \circ g)(x^2)$

$$= h\{g(x^2)\} = h(\tan x^2) = \log(\tan x^2).$$

$\therefore \quad \{h \circ (g \circ f)\}\sqrt{\frac{\pi}{4}} = \log\left(\tan\frac{\pi}{4}\right) = \log 1 = 0.$

23. Dom $(g \circ f) = \text{dom}(f) = \{1, 3, 4\}$.

$(g \circ f)(1) = g\{f(1)\} = g(2) = 3, (g \circ f)(3) = g\{f(3)\} = g(5) = 1$

$(g \circ f)(4) = g\{f(4)\} = g(1) = 3$

$\therefore \quad g \circ f = \{(1,3), (3,1), (4,3)\}.$

24. $f(x)$ is defined only when $9 - x^2 \geq 0 \Rightarrow x^2 \leq 9 \Rightarrow -3 \leq x \leq 3.$

$\therefore \quad \text{dom}(f) = [-3, 3].$

25. $f(x)$ is defined when $x - 4 \neq 0$ and $\frac{x-1}{x-4} \geq 0$

$\Rightarrow \quad x \neq 4$ and $(x \geq 4$ or $x \leq 1) \Rightarrow (x > 4$ or $x \leq 1)$

$\Rightarrow \quad \text{dom}(f) = (-\infty, 1] \cup (4, \infty).$

26. $f(x)$ is defined only when $(x^2 - 1) \geq 0$ and $(x - 1) > 0$

$\Rightarrow \quad (x-1)(x+1) \geq 0$ and $(x-1) > 0 \Rightarrow x + 1 \geq 0$ and $x - 1 > 0 \Rightarrow x > 1$

$\therefore \quad \text{dom}(f) = (1, \infty).$

27. $f(x)$ is not defined when $(x^2 - 1) = 0$, i.e., when $(x-1)(x+1) = 0$,

i.e., when $x = 1$ or $x = -1$.

$\therefore \quad \text{dom}(f) = R - \{1, -1\}.$

28. $\frac{\sin^{-1} x}{x}$ is defined only when $x \neq 0$ and $x \in [-1, 1]$.

$\therefore \quad \text{dom}(f) = [-1, 1] - \{0\}.$

29. $\sin^{-1} 2x$ is defined only when $2x \in [-1, 1] \Rightarrow x \in \left[\frac{-1}{2}, \frac{1}{2}\right].$

30. $\cos^{-1}(3x - 1)$ is defined only when $(3x - 1) \in [-1, 1]$

$\Rightarrow \quad 3x \in [0, 2] \Rightarrow x \in \left[0, \frac{2}{3}\right] \Rightarrow \text{dom}(f) = \left[0, \frac{2}{3}\right].$

31. $f(x)$ is defined only when $\cos x \geq 0$

$\Rightarrow \quad x$ lies in 1st or 4th quadrant

$\Rightarrow \quad \text{dom}(f) = \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right].$

32. $f(x)$ is defined only when $\log(2x-x^2)\geq 0$

$\Rightarrow\ (2x-x^2)\geq 1 \Rightarrow (1+x^2-2x)\leq 0 \Rightarrow (1-x)^2\leq 0 \Rightarrow (1-x)=0 \Rightarrow x=1.$

$\therefore$ dom $(f)=\{1\}$.

33. $f(x)=x^2$ is clearly defined for each $x\in R$. So, dom $(f)=R$.

$y=x^2 \Rightarrow x=\pm\sqrt{y}$.

When $y<0$, there is no real value of x. So, $y\geq 0$.

$\therefore$ range $(f)=R^+$.

34. $f(x)=x^3$ is defined for each $x\in R$. So, dom $(f)=R$.

For each $y\in R$, $y^{1/3}\in R$ and so $x=y^{1/3}$ is real.

$\therefore$ range $(f)=R$.

35. Let $f(x)=g(x)+h(x)$, where $g(x)=\log(1-x)$ and $h(x)=\sqrt{x^2-1}$.

$g(x)$ is defined only when $1-x>0 \Rightarrow x<1$. So, dom $(g)=(-\infty,1)$.

$h(x)$ is defined only when $x^2-1\geq 0 \Rightarrow x\geq 1$ or $x\leq -1$.

$\therefore$ dom $(h)=(-\infty,-1]\cup[1,\infty)$.

$\therefore$ dom $(f)=$ dom $(g)\cap$ dom $(h)=(-\infty,-1]$.

36. $y=\dfrac{1}{(1-x^2)} \Rightarrow x=\sqrt{1-\dfrac{1}{y}}$.

Clearly, x is not defined when $y=0$ or $1-\dfrac{1}{y}<0$, i.e., $y=0$ or $y<1$.

$\therefore$ range $(f)=[1,\infty)$.

37. $y=\dfrac{x^2}{(1+x^2)} \Rightarrow x=\sqrt{\dfrac{y}{1-y}}$.

Clearly, x is defined only when $\dfrac{y}{(1-y)}\geq 0$, and $(1-y)\neq 0$, i.e., when $0\leq y<1$.

So, range $(f)=[0,1)$.

38. $y=\dfrac{x^2+1}{x} \Rightarrow x^2-xy+1=0 \Rightarrow x=\dfrac{y\pm\sqrt{y^2-4}}{2}$.

x is defined when $(y^2-4)\geq 0 \Rightarrow y^2\geq 4 \Rightarrow y\geq 2$ or $y\leq -2$.

$\therefore$ range $(f)=(-\infty,-2]\cup[2,\infty)$.

39. Clearly, $a^x>0$ whatever may be the value of x.

$\therefore$ range $(f)=(0,\infty)$.